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NUMERICAL SOLUTIONS OF THE NONLINEAR AXISYMMETRIC EQUATIONS FOR SHELLS OF REVOLUTION

TECHNICAL REPORT

by

JOHN F. MESCALL

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ABSTRACT

A numerical procedure for the solution of the nonlinear equations governing the large axisymmetric deflections of thin shells of revolution is presented and applied both to the complete equations due to Reissner and to the simpler equations to which these reduce for small-finite angle changes. Global solutions extending into the postbuckled range are shown to be considerably more complicated than expected. The character of the global solution is also shown to be quite sensitive to boundary conditions imposed. A comparison of the results obtained from the complete equations and the small-finite deflection equations reveals a very close agreement through the entire load-deflection history.

CONTENTS

NOMENCLATURE

ϕ, ϕ_o = slope of meridian on deformed, undeformed middle surface

h = shell thickness

$r_o(\xi), z_o(\xi)$ = horizontal and vertical coordinates of undeformed middle surface

α_o^2 = $(r_o')^2 + (z_o')^2$, where primes refer to differentiation with respect to the independent variable, ξ.

ν, E = Poisson's ratio, Young's modulus

C = $1/(Eh)$

D = $Eh^3/12(1-\nu^2))$

V, H = vertical, horizontal stress resultants

ψ = $r_o H$ = stress function

β = $(\phi_o - \phi)$ = rotation of meridional tangent

p = uniform pressure, psi

$\bar{p}$ = nondimensionalized pressure

p_H, p_V = horizontal, vertical components of uniform pressure

P = concentrated load at apex of spherical shell

P^* = $aP/(2\pi Eh^3)$ = nondimensional load parameter

N_ξ, N_θ = meridional, circumferential stress resultants

M_ξ, M_θ = meridional, circumferential stress couples

u, w = horizontal, vertical deflections

$w(o)$ = vertical deflection at apex of sphere

k = mesh spacing

a = radius of spherical shell

ξ_o = half-angle opening of spherical shell

m^4 = $12(1-\nu^2)$

λ^2 = $m^2\xi_o^2\, a/h$ = nondimensional geometric parameter for shallow spherical shell.

INTRODUCTION

The objective of this paper is the description and application of a numerical procedure for the solution of the nonlinear differential equations governing the finite axisymmetric deformation of thin shells of revolution as presented by Reissner.[1] The procedure is applied both to the complete equations (I and II of Reference 1), valid for arbitrarily large deflections consistent with small strains, as well as to the simpler set of equations (III and IV of Reference 1) valid for small-finite angle changes. The latter set of equations (or their equivalent) has formed the basis for many previous investigations in this area.[2-6] Thurston[7] has studied the complete equations but does not report on a comparison of the results of the two sets of equations. The present study consists essentially of an application of Newton's method[8] to obtain a system of linear correctional equations for an initial approximate solution and the employment of a Gaussian elimination procedure[9] for the solution of the finite difference equivalent of this linear system. Archer,[4] Wilson and Spier,[10] and others have previously employed a Gaussian elimination procedure in this problem area, while Thurston[5] has utilized Newton's method for the case of uniform pressure on a clamped spherical cap. The specific advantage provided by the procedure discussed in this paper is that it more completely characterizes the solutions. In particular, it permits development of global solutions which are shown to be continuous from the small deflections encountered in the prebuckled state through the large deflections of the postbuckled state. The continuous character of these solutions is not only interesting from a theoretical point of view but is also useful in permitting one to develop a complete load-deflection history for a new geometry and loading condition without having *a priori* estimates of the solution in either the prebuckled or the postbuckled state.

To illustrate this latter point, we consider the problems of a clamped spherical shell under uniform pressure and that of a spherical shell under a concentrated load at the apex, with a clamped edge and with an unrestrained edge. New results are presented for these problems in the so-called postbuckled range.

BASIC EQUATIONS

Reissner[1] has given the equations governing the finite axisymmetric deflections of thin shells of revolution in the form

$$(\phi-\phi_o)'' + (F_o'/F_o)(\phi-\phi_o)' - (\alpha_o/r_o)^2(\cos\phi)(\sin\phi - \sin\phi_o)$$

$$+ \nu\,(\alpha_o/r_o)\left[(\cos\phi - \cos\phi_o)\phi_o' + (D'/D)(\sin\phi - \sin\phi_o)\right]$$

$$= \alpha_o^2/(r_oD)\,\left[\psi\sin\phi - r_oV\,\cos\phi\right] \tag{1}$$

$$\psi'' + (G_o'/G_o)\psi' - \left[(\alpha_o\cos\phi/r_o)^2 - \nu(\alpha_o/r_o)(\phi'\sin\phi + C'\cos\phi/C)\right]\psi$$

$$= (\alpha_o^2C/r_o)(\cos\phi - \cos\phi_o) + \nu\alpha_o(\sin\phi)(r_oV)'/r_o$$

$$+ \left[(\alpha_o/r_o)^2\cos\phi\sin\phi + \nu(\alpha_o/r_o)(\phi'\cos\phi - C'\sin\phi/C)\right]\,r_oV$$

$$- (\alpha_o/r_o)(r_o^2\,p_H)' - \left[\nu(\alpha_o/r_o)^2\cos\phi - (\alpha_oC')/(r_oC)\right]\,r_o^2\,p_H \tag{2}$$

where $$F_o = (r_oD/\alpha_o), \qquad G_o = r_o/(\alpha_oC),$$

where primes denote differentiation with respect to the independent variable ξ, and where subscript zero refers to the value of the subscripted quantity before deformation (see Figure 1).

The appropriate stress and moment resultants and displacements are expressed in terms of ϕ and ψ by the relations

$$r_oV = -\int r_op_V\,\alpha_o d\xi$$

$$r_oN_\xi = \psi\cos\phi + r_oV\sin\phi$$

$$r_oQ = -\psi\sin\phi + r_oV\cos\phi$$

$$\alpha_oN_\theta = \psi' + r_o\alpha_op_H \tag{3}$$

$$M_\xi = -D\left[(\phi-\phi_o)'/\alpha_o + \nu(\sin\phi - \sin\phi_o)/r_o\right]$$

$$M_\theta = -D\left[(\sin\phi - \sin\phi_o)/r_o + \nu(\phi-\phi_o)'/\alpha_o\right]$$

$$u = r_o(N_\theta - \nu N_\xi)/C$$

$$w = \int\left[(\sin\phi - \sin\phi_o) + \sin\phi\,(N_\xi - \nu N_\theta)/C\right]\alpha_o d\xi.$$

For small-finite deflections ($\beta^2 \equiv (\phi_o-\phi)^2 \ll 1$), Equations 1 and 2 simplify to

$$F_o\beta'' + F_1\beta' + F_2\beta + F_3\psi = \Gamma_1 \tag{4}$$

$$G_o\psi'' + G_1\psi' + G_2\psi + G_3\beta = \Gamma_2 \tag{5}$$

where $\qquad F_1 = F_o' \qquad G_1 = G_o'$

$$F_2 = -(r_o')^2 D/(r_o\alpha_o) + \nu(r_o'D/\alpha_o)'$$

$$G_2 = -(r_o')^2/(r_o\alpha_o C) - \nu(r_o'/\alpha_o C)'$$

$$F_3 = \alpha_o \sin\phi_o \qquad G_3 = -F_3$$

$$\begin{aligned}\Gamma_1 = {} & \left[(3r_o'z_o'D)/(2r_o\alpha_o) - \nu(z_o'D/2\alpha_o)'\right]\beta^2 \\ & + \alpha_o r_o V\cos\phi_o + \alpha_o\beta\left[\psi\cos\phi_o + r_o V\sin\phi_o\right]\end{aligned} \tag{6}$$

$$\begin{aligned}\Gamma_2 = {} & \left[(2z_o'r_o')/(r_o\alpha_o C) + \nu(z_o'/\alpha_o C)'\right]\beta\psi + \nu z_o'\beta'\psi/(\alpha_o C) \\ & - \alpha_o\beta^2\cos\phi_o/2 + \left[r_o'z_o'/(r_o\alpha_o C) + \nu(z_o'/\alpha_o C)'\right](r_o V) \\ & + \nu(z_o'/\alpha_o C)(r_o V)' + \left[(z_o'^2 - r_o'^2)/(\alpha_o r_o C) - \nu(r_o'/\alpha_o C)'\right](r_o V)\beta \\ & - \nu r_o'(\beta r_o V)'/(\alpha_o C) - (r_o^2 p_H)'/C - \left[\nu(r_o' + \beta z_o') - r_o C'/C\right](r_o p_H/C)\end{aligned}$$

The corresponding simplification of Equations 3 will not be reproduced here in the interest of brevity.

METHOD OF SOLUTION

As mentioned, the numerical technique employed in this report combines the standard Gaussian elimination procedure with the classical Newton iteration method. The former substantially reduces the demands made upon a digital computer with respect to both memory and speed. The latter improves convergence significantly and, when coupled with a fairly direct method for obtaining starting solutions, permits the development of solutions in the so-called postbuckled range.

Details of the procedure are perhaps most compactly presented in terms of the small-finite deflection equations. Extension to the more general equations follows easily. The essence of the Newtonian iterative scheme as outlined in Reference 8 consists in replacing the nonlinear differential equations by a sequence of linear differential equations. Specifically, a correction $(\delta\beta_j, \delta\psi_j)$ to the approximate solution (β_j, ψ_j) is sought according to the prescription

$$
\begin{aligned}
\beta &= \beta_j + \delta\beta_j \\
\psi &= \psi_j + \delta\psi_j
\end{aligned}
\qquad (7)
$$

where (β, ψ) is the actual solution. Inserting (7) into (4) and (5) and omitting terms in Γ_1, Γ_2 which are negligible, the result may be written

$$
\begin{aligned}
&F_o\, \delta\beta_j'' + F_1 \delta\beta_j' + (F_2 - \alpha_o \psi_j \cos\phi_o)\, \delta\beta_j + (F_3 - \alpha_o \beta_j \cos\phi_o)\, \delta\psi_j \\
&\qquad = \Gamma_1^* = \Gamma_1 - \{F_o \beta_j'' + F_1 \beta_j' + F_2 \beta_j + F_3 \psi_j\} \\
&G_o\, \delta\psi_j'' + G_1 \delta\psi_j' + G_2 \delta\psi_j + (G_3 + \alpha_o \beta_j \cos\phi_o)\, \delta\beta_j \\
&\qquad = \Gamma_2^* = \Gamma_2 - \{G_o \psi_j'' + G_1 \psi_j' + G_2 \psi_j + G_3\, \beta_j\}.
\end{aligned}
\qquad (8)
$$

In Equations 8 we have omitted nonlinear terms in $\delta\beta_j$ and $\delta\psi_j$, since for an assumed solution (β_j, ψ_j) reasonably close to the correct solution these terms are small compared to the linear terms. An iterative procedure is now adopted in which Equations 8 are solved for $(\delta\beta_j, \delta\psi_j)$ and a new approximate solution $(\beta_{j+1}, \psi_{j+1})$ is obtained from

$$\begin{aligned} \beta_{j+1} &= \beta_j + \delta\beta_j \\ \psi_{j+1} &= \psi_j + \delta\psi_j . \end{aligned} \tag{9}$$

If the correction terms $(\delta\beta_j, \delta\psi_j)$ approach zero as the number of iterations increase, then the approximate solution obtained by this iterative process approaches an exact solution of the original equations. For the solution of Equations 8, we replace derivatives by simple central differences over a mesh of n points with spacing k and obtain

$$A_i \delta T_{i-1} + B_i \delta T_i + C_i \delta T_{i+1} = D_i \tag{10}$$

$$i = 2,3,\dots\; n-1$$

where, in the compact notation employed by Archer,[4]

$$T_i = \begin{pmatrix} \beta \\ \psi \end{pmatrix}_i \qquad \delta T_i = \begin{pmatrix} \delta\beta \\ \delta\psi \end{pmatrix}_i \tag{11}$$

$$A_i = \begin{pmatrix} F_o - kF_1/2 & 0 \\ 0 & G_o - kG_1/2 \end{pmatrix}_i , \quad C_i = \begin{pmatrix} F_o + kF_1/2 & 0 \\ 0 & G_o + kG_1/2 \end{pmatrix}_i$$

$$B_i = \begin{pmatrix} -2F_o + k^2(F_2 - \alpha_o(\cos\phi_o)\psi) & k^2(F_3 - \alpha_o(\cos\phi_o)\beta) \\ k^2(G_3 + \alpha_o(\cos\phi_o)\beta) & -2G_o + k^2G_2 \end{pmatrix}_i , \quad D_i = \begin{pmatrix} k^2\Gamma_1^* \\ k^2\Gamma_2^* \end{pmatrix}_i \tag{12}$$

If the original boundary conditions are formulated as

$$\begin{aligned} B_1T_1 + C_1T_2 &= D_1 \\ A_nT_{n-1} + B_nT_n &= D_n \end{aligned} \tag{13}$$

then the boundary conditions for the modified problem become

$$B_1\delta T_1 + C_1\delta T_2 = D_1 - B_1T_1 - C_1T_2 = D_1^* = 0$$

$$A_n\delta T_{n-1} + B_n\delta T_n = D_n - A_nT_{n-1} - B_nT_n = D_n^* = 0 \ . \qquad (14)$$

The process of Gaussian elimination very efficiently inverts this system of equations. This process may be compactly summarized by the relations[4]

$$W_1 = B_1 \qquad S_1 = W_1^{-1}D_1 \qquad R_1 = W_1^{-1}C_1$$

$$\begin{aligned} W_i &= B_i - A_iR_{i-1} & S_i &= W_i^{-1}(D_i - A_iS_{i-1}) \\ R_i &= W_i^{-1}C_i & i &= 2,3,\dots\ n-1 \\ W_n &= B_n - A_nR_{n-1} & S_n &= W_n^{-1}(D_n - A_nS_{n-1}) \end{aligned} \qquad (15)$$

$$T_n = S_n \qquad T_i = S_i - R_iT_{i-1} \qquad i = n-1,\dots\ 1.$$

A detailed description of the selection of starting values and of the procedure for obtaining solutions in the postbuckled regions is best given with reference to a specific example and is postponed until the next section.

Reissner's complete equations may be treated in essentially the same fashion, with terms such as cos ϕ replaced by

$$\cos\phi \equiv \cos(\phi_o - \beta) = \cos(\phi_o - \beta_i - \delta\beta_i)$$

$$= \cos(\phi_o - \beta_i) + \delta\beta_i \sin(\phi_o - \beta_i) \qquad (16)$$

and

$$\sin\phi = \sin(\phi_o - \beta_i) - \delta\beta_i \cos(\phi_o - \beta_i).$$

The finite difference simulation of the complete equations then has the same form as that for the small-finite deflection equations. The modifications of the definitions for F_K, G_K are not presented here in the interest of brevity, and, also, in view of the outcome of the comparison to be made between the results of the two sets of equations.

NUMERICAL RESULTS

a. Concentrated Load on Spherical Shell

Consider first the case of a spherical segment with a concentrated load P at the apex and an unrestrained edge. For this problem the equations of the middle surface of the shell are taken as

$$r_o = a \sin \xi, \qquad z_o = -a \cos \xi \tag{17}$$

while

$$p_H = p_V = 0, \qquad (r_o V) = P/2\pi. \tag{18}$$

Convenient nondimensional geometric and load parameters are defined as

$$\lambda^2 = m^2 \xi_o^2 \, a/h \quad \text{and} \quad P^* = aP/(2\pi E h^3). \tag{19}$$

We observe that although it is not necessary to confine attention to shallow shells, nonetheless, due to the nature of the load, this is the most interesting area of application. If deep shells are being considered, the definition of λ^2 involves $\sin^2(\xi_o)$ rather than ξ_o^2 . Appropriate boundary conditions are

$$\begin{aligned} \beta = \psi = 0 \qquad & \text{at} \quad \xi = 0 \\ M_\xi = \psi = 0 \qquad & \text{at} \quad \xi = \xi_o. \end{aligned} \tag{20}$$

With this information one may proceed as follows: For a given value of λ and P^* the linear solution is first obtained by suppressing nonlinear terms in Γ_1 and Γ_2. (In this connection, P^* is initially chosen small enough that the linear solution is reasonably appropriate). The linear solution is used as an initial estimate of the nonlinear solution (β_o, ψ_o) for the same P^*, and Equations 8 are solved for the corrections ($\delta\beta_o$, $\delta\psi_o$) to this initial estimate. The corrected set of solution values (β_1, ψ_1) is used to obtain a new set of corrections ($\delta\beta_1$, $\delta\psi_1$), and the process is repeated until the solution converges to a specified degree of accuracy. In this connection, it should be noted that as a convergence criterion we employed the requirement that for the m^{th} iteration and for each point i on the mesh

$$\frac{(\beta_i)_m - (\beta_i)_{m-1}}{(\beta_i)_m} < 0.001\ , \qquad \frac{(\psi_i)_m - (\psi_i)_{m-1}}{(\psi_i)_m} < 0.001. \tag{21}$$

When satisfactory convergence has been achieved for a given value of P^*, stresses and displacements along a shell meridian may be calculated. Then P^* is incremented and the previously converged solution for the lower value of P^* is used to start the iteration for the higher value of P^*. In this manner curves of load versus deflection are obtained for given values of λ. It is a relatively direct matter to move along a branch of the load-deflection curve until a local maximum (or minimum) is reached. At such a point, the following simple procedure was found to be adequate for obtaining starting solutions on the neighboring (continuous) branch. For a value of P^* slightly below the maximum (or above the minimum), take as a starting set of β and ψ a multiple $(\theta_1\beta,\ \theta_2\psi)$ of the solution found for P^* on the preceding branch. A very small amount of numerical experimentation is sufficient to find constant values of θ_1 and θ_2 which produce a convergent solution on the new branch. These values of θ_1 and θ_2 depend on λ^2 and P^*, but generally, $1.0 < (\theta_1, \theta_2) < 3.0$ for branches moving to the right (larger deflections), while $0.50 < (\theta_1, \theta_2) < 1.0$ for branches moving to the left (smaller deflections). It is now a simple matter to move along the new branch until another local minimum (or maximum) is found.

For the specific problem under discussion, the load deflection curves for small λ (say $\lambda^2 \leq 14$) are found to be monotonically increasing. As λ increases, local maxima and minima emerge but the curves still retain the generic shape frequently associated with (or postulated for) the mechanism of buckling (see Figure 2). Our objective in this report is not so much the prediction of buckling loads for this problem (this was done in Reference 11, where excellent agreement with the experimental results of Evan-Iwanowski, et al,[12] was demonstrated), but is concerned rather with the observation that for higher λ values the character of the load-deflection curve changes considerably. For example, in Figures 3a, b, and c, we present numerical results for $\lambda^2 = 64$, 81, and 144.

To emphasize the dependence of the solution upon boundary conditions we present in Figure 4 the load-deflection curve obtained for the same problem, with $\lambda^2 = 144$ but with a clamped edge and, therefore, the boundary conditions

$$\begin{aligned} \beta = \psi = 0 \qquad & \text{at} \quad \xi = 0 \\ \beta = u = 0 \qquad & \text{at} \quad \xi = \xi_o. \end{aligned} \tag{22}$$

The difference in behavior of the solution produced by this change in the boundary condition is rather surprising in view of the loading. We observe that there is also experimental evidence of a marked difference in the behavior of clamped versus unrestrained spherical caps under concentrated loads. Evan-Iwanowski, et al,[12] concluded on the basis of detailed experimental studies that such a clamped shell exhibits the load deformation pattern shown in Figure 4 and, therefore, does not buckle.

Finally, we observe that the numerical results obtained by using the complete equations differed only very slightly from those obtained from the small-finite equations. The overall character of the two sets of results is the same, i.e., the load-deflection curves have the same continuous character and the same number of branches. Numerically, the typical deviation along a branch of the curves is of the order of one percent, even for the final post-buckled branch where maximum discrepancy is to be expected. The greatest deviations occur near the maxima or minima of a branch, but even there the discrepancy is so slight that the difference between the load-deflection curves is barely discernible on a graph of reasonable scale.

b. Uniform Pressure on Spherical Cap

Turning next to the case of uniform pressure p on a clamped spherical shell, we have for loading conditions

$$p_H = p \sin \xi \qquad p_V = -p \cos \xi \qquad r_o V = r_o^2 \, p/2 \tag{23}$$

where

$$r_o = a \sin \xi \qquad z_o = -a \cos \xi. \tag{24}$$

Nondimensional geometric and load parameters are

$$\lambda^2 = m^2 \xi_o^2 \, a/h \qquad \text{and} \qquad \bar{p} = -m^2 a^2 p/(4Eh^2). \tag{25}$$

We shall also make use of the "average deflection" parameter ρ, defined as

$$\rho = m^2 \bar{w}/h$$

where

$$\bar{w} = (2/R_o^2) \int_o^{R_o} r \, w \, dr \tag{26}$$

and

$$R_o = a \sin \xi_o .$$

Boundary conditions for the clamped spherical shell are

$$\begin{aligned} \beta = \psi = 0 \qquad & \text{at } \xi = 0 \\ \beta = u = 0 \qquad & \text{at } \xi = \xi_o . \end{aligned} \tag{27}$$

Typical load-deflection curves obtained for this problem are illustrated in Figures 5 through 8. These curves bring out a gradual transition from the simpler behavior for $\lambda = 5$ through a more elaborate behavior for $\lambda = 8$, a return to a relatively simple pattern for $\lambda = 12$, and finally, the emergence of the more intricate behavior again at $\lambda = 20$.

As in the case of the concentrated load, use of the complete equations produced no essential change in the qualitative behavior of the solution, viz., the load-deflection curves are continuous and have the same number of branches as do the solutions of the small-finite equations. Quantitatively, the agreement is again very good, with the greatest discrepancies being of the order of a few percent and occurring at the maxima or minima of the curves. The agreement persists even into the final branch of the curves, which corresponds to a nearly inverted shallow shell.

DISCUSSION OF RESULTS

The present numerical results demonstrate the existence of a large number of distinct equilibrium positions for a given load and for certain ranges of the geometric parameter. The existence of these multiple solutions is also shown to be dependent to a great extent upon the boundary conditions but to be independent of the employment of the complete equations rather than the small-finite deflection equations. Recently (during the preparation of this paper, in fact), it came to the writer's attention that Anselone, Bueckner, Johnson, and Moore,[13,14] have studied the large deflections of a clamped shallow spherical cap under uniform pressure and, employing a completely different technique from the one described here, have obtained results which exhibit the same continuous character and which agree quite well numerically with those presented here. We note also that Thurston[5] and Keller and Reiss[6] have pointed out the possibility of multiple equilibrium configurations for pressure loading on a clamped spherical cap, although their results do not bring out the continuity of the solutions.

The increasing complexity in the structure of the solutions as λ increases brings out some interesting features. In particular, the present results demonstrate (for the clamped, pressurized spherical cap and the unrestrained spherical cap under concentrated load) the existence of a number of bifurcation points in the axisymmetric solution. We observe that although two distinct equilibrium positions are possible at such a point, the distributions of displacements along a meridian were found to be quite dissimilar, and a change from one path to the other would involve finite (rather than infinitesimal) changes in deflection. It is interesting, therefore, to note that the shell under concentrated load apparently ignores these bifurcation points since the load-deflection curves obtained experimentally by Evan-Iwanowski et al,[12] are in very good agreement with the branch producing the first maximum on the continuous curve.

In the case of uniform pressure, it is interesting to observe that if one defines the critical pressure as that corresponding to the first maximum on the load-deflection curve, the results obtained in the present study agree very closely with the four most widely accepted sets of results for the axisymmetric treatment of this problem.[2-5] Furthermore, our results for the minimum pressure in the postbuckled range agree quite closely with those obtained by Thurston,[5] who was able to obtain results in this zone without establishing the continuity of the global solution. One clear advantage of having the complete solution is the following: As Budiansky[2] points out, referring to cases where the maxima were not located with certainty: "Strictly speaking, it must be conceded that the upper bounds in these cases have not been rigorously established, since it is conceivable that failure to converge might be due to some unknown cause other than the nonexistence of an adjacent equilibrium position." The present results make it clear that the results of References 2 through 5 are actually the first maxima on the load-deflection curves. They also reinforce (though any further reinforcement is hardly necessary in view of the results of Reference 15) Budiansky's assertion that one must include asymmetric effects in the study of the buckling of spherical shells. We conjecture here that the axisymmetric global solutions may prove useful in the more difficult problem area of large nonsymmetric deformations by providing basic states about which to expand, particularly in the post-buckled zone.

It is considered worth mentioning that strain and potential energies were computed for the various deformation states and the so-called energy criteria of buckling (i.e., constant volume and constant pressure buckling) were applied. The results again agreed quite closely with those reported by Thurston,[7] and are not reported here in the interest of brevity. However, we observe, strictly speaking, that in order to employ such an energy criterion one should have the complete global solution available in order that the energy levels of all competing equilibrium states be compared.

Finally, we remark that the degree of compatibility between the results of the complete equations and the small-finite deflection equations is somewhat surprising since the latter contain only the simplest type of nonlinearity, whereas, the nonlinearity in the complete equations is more complex. The analytical and practical advantages offered by the simpler equations are sufficiently large - for example, the computer program for the simpler set takes only one-third as long to run—that exploitation should be made of this agreement whenever possible.

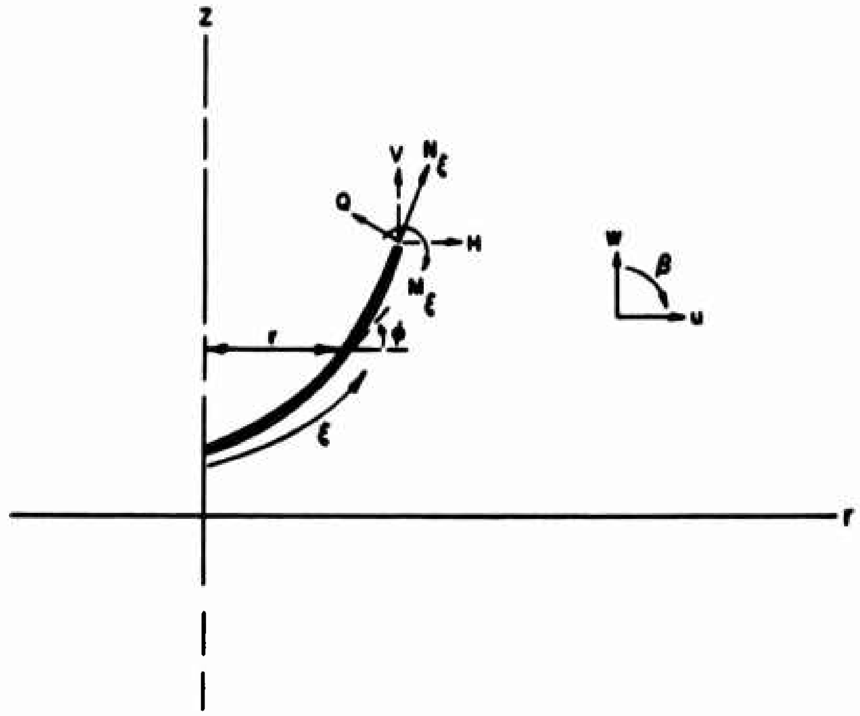

Figure 1. GEOMETRY

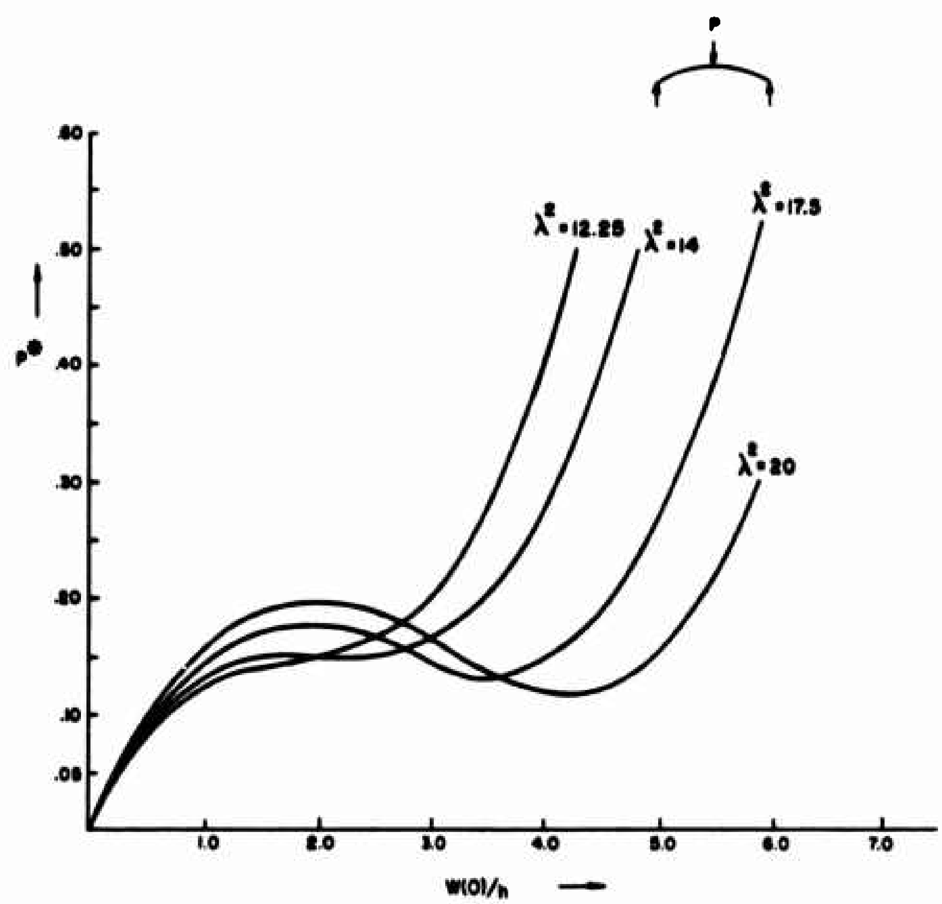

Figure 2. LOAD-DEFLECTION CURVES (UNRESTRAINED EDGE)

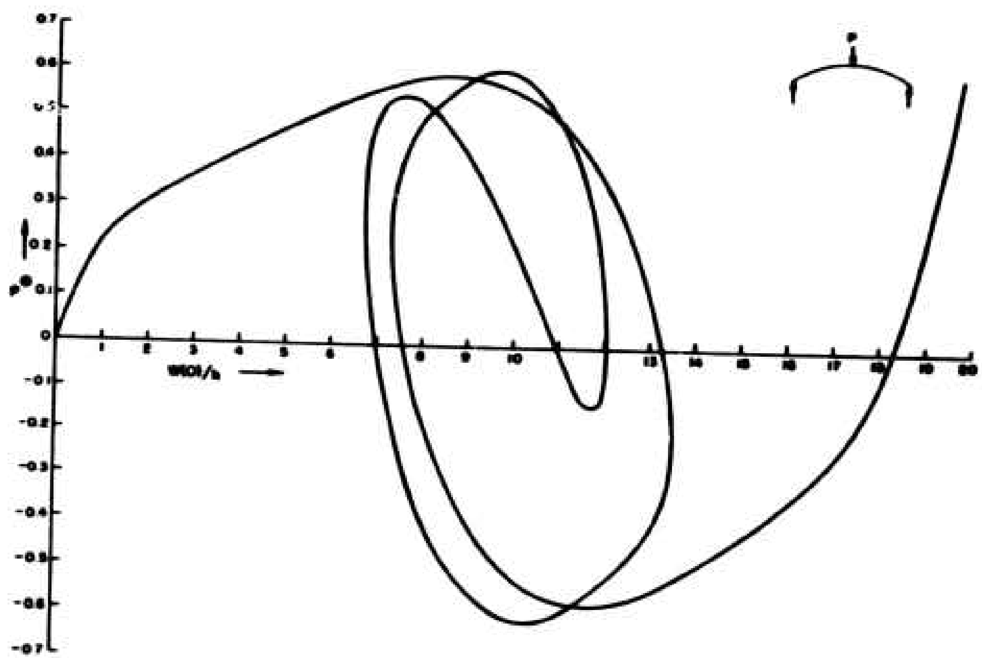

Figure 3a. LOAD-DEFLECTION CURVE (UNRESTRAINED EDGE; $\lambda^2 = 64$)

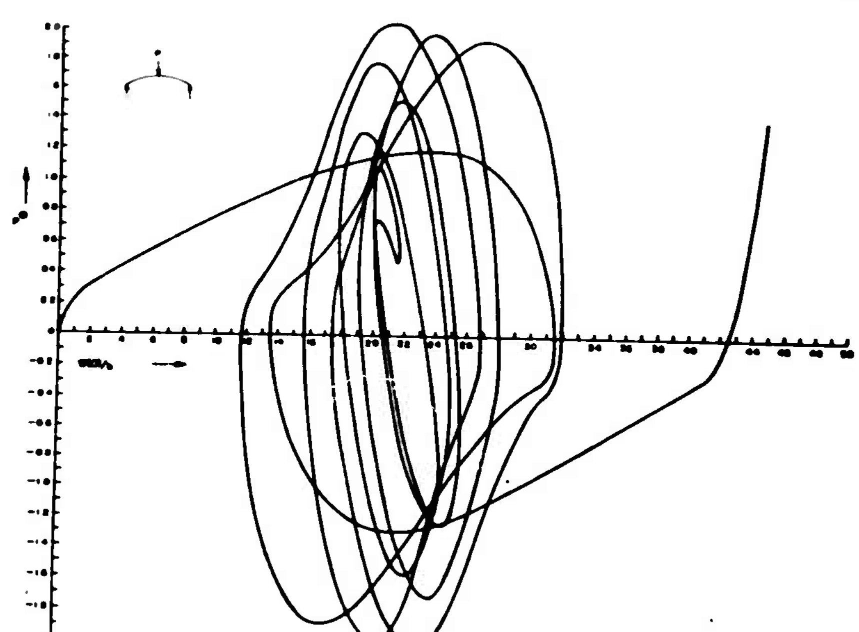

Figure 3c. LOAD-DEFLECTION CURVE (UNRESTRAINED EDGE: $\lambda^2 = 144$)

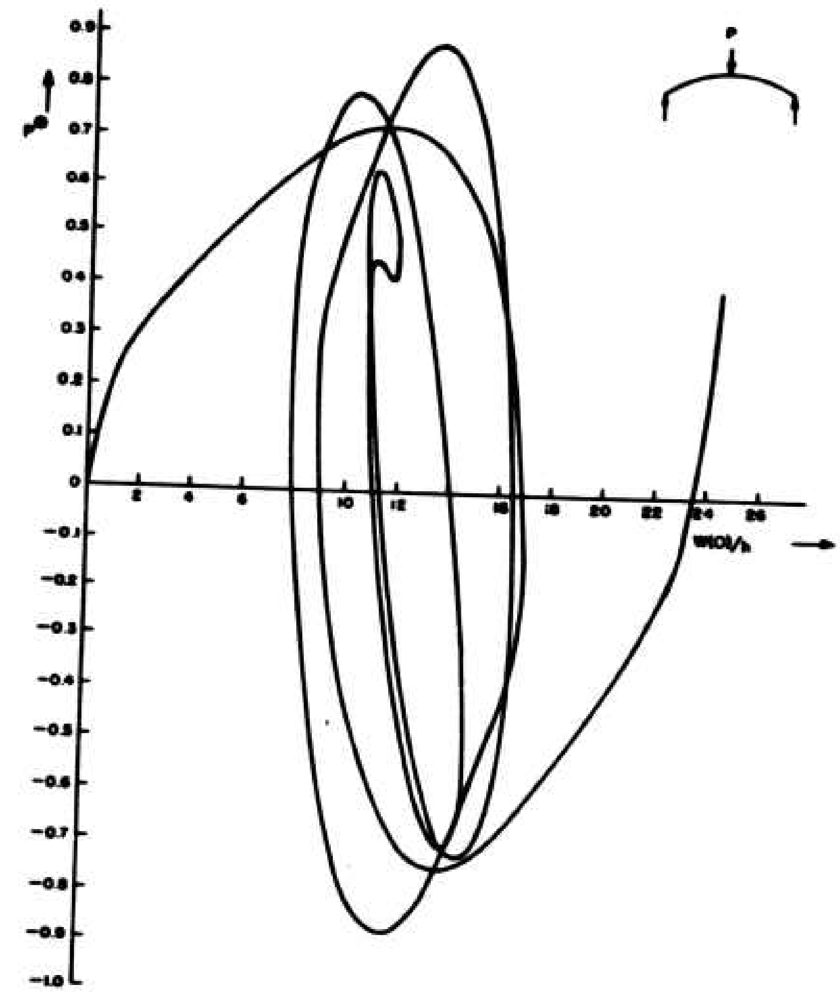

Figure 3b. LOAD-DEFLECTION CURVE
(UNRESTRAINED EDGE; $\lambda^2 = 81$)

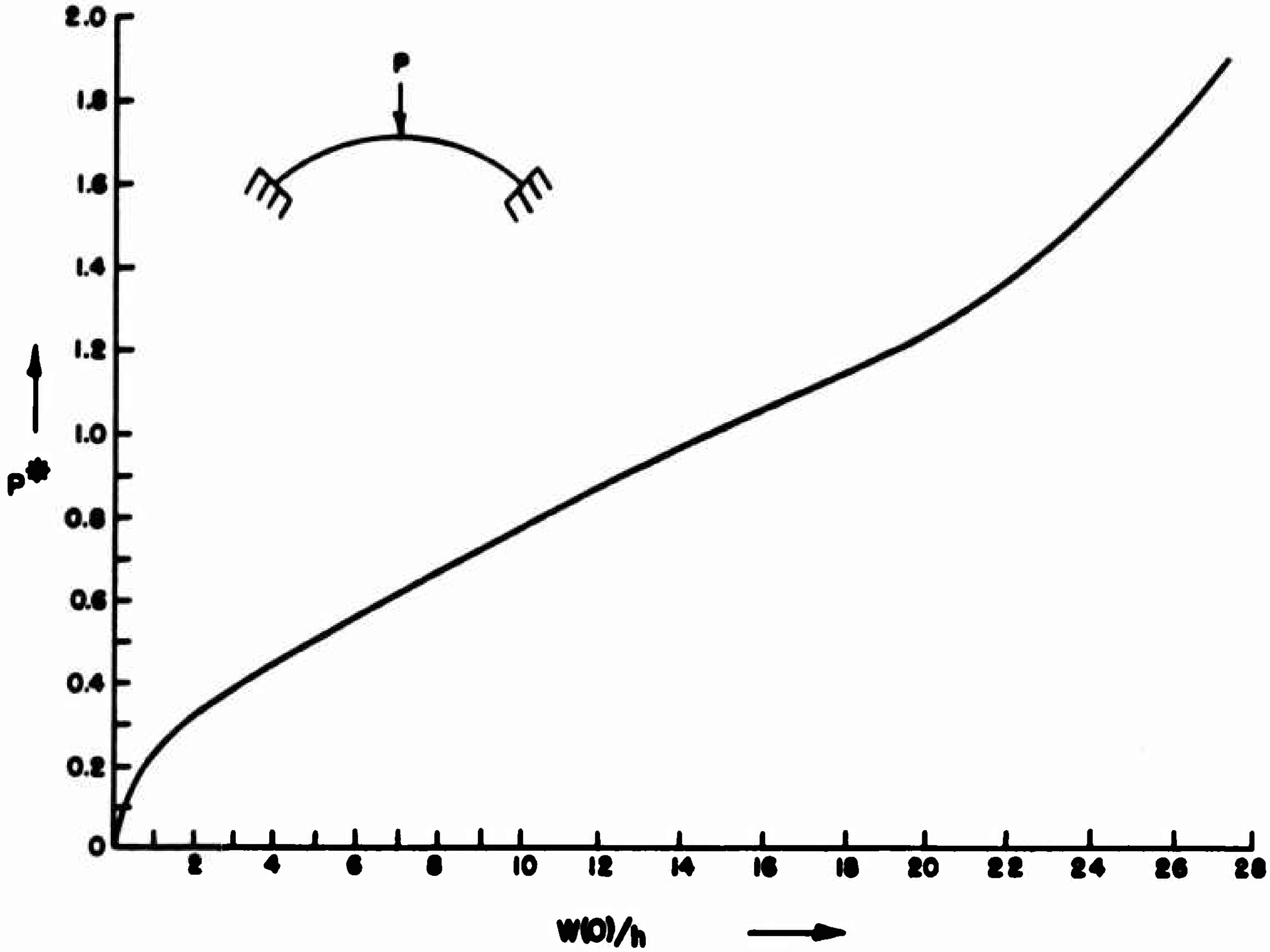

Figure 4. LOAD-DEFLECTION CURVE (CLAMPED EDGE; $\lambda^2 = 144$)

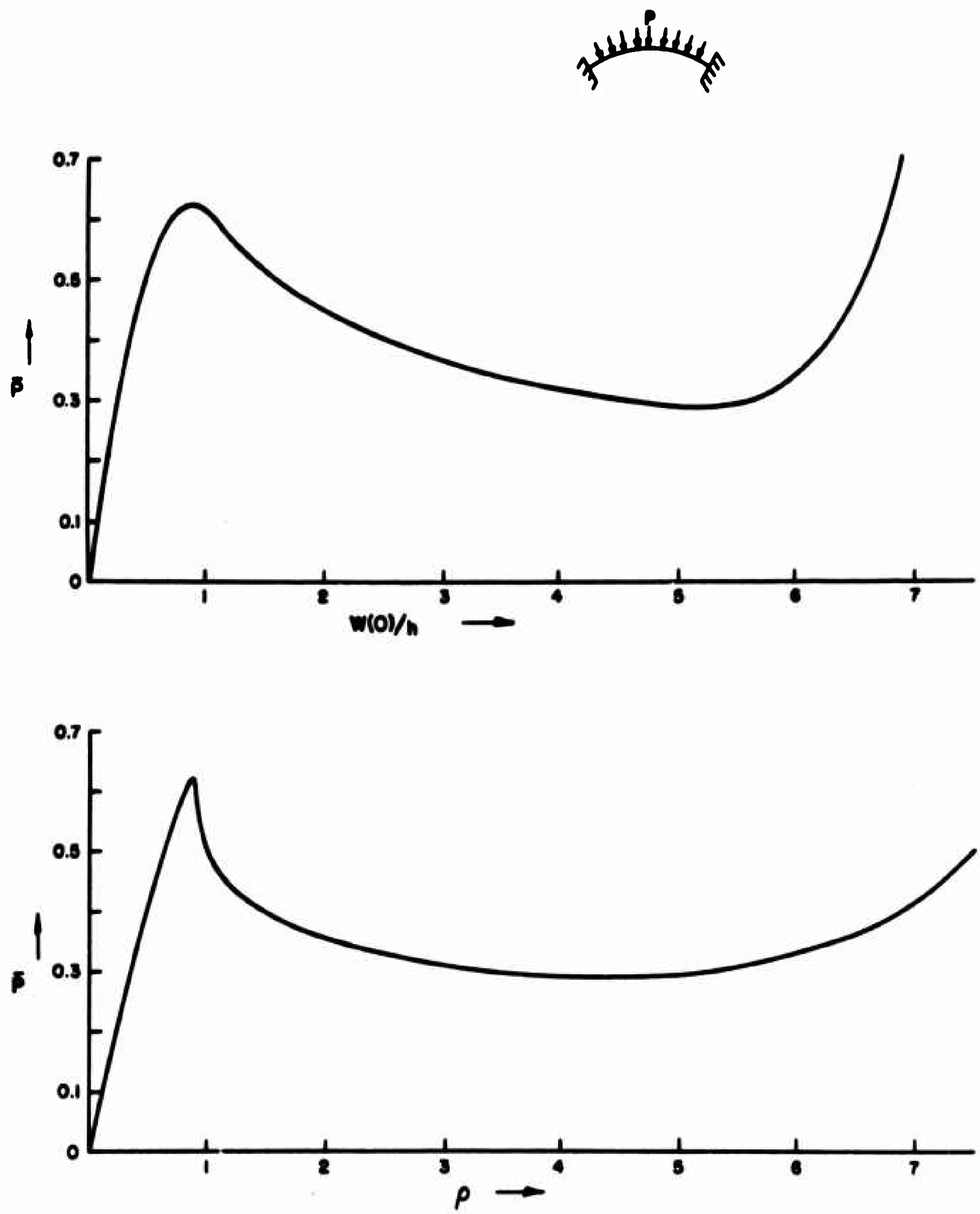

Figure 5. LOAD-DEFLECTION CURVE (CLAMPED EDGE; $\lambda^2 = 25$)

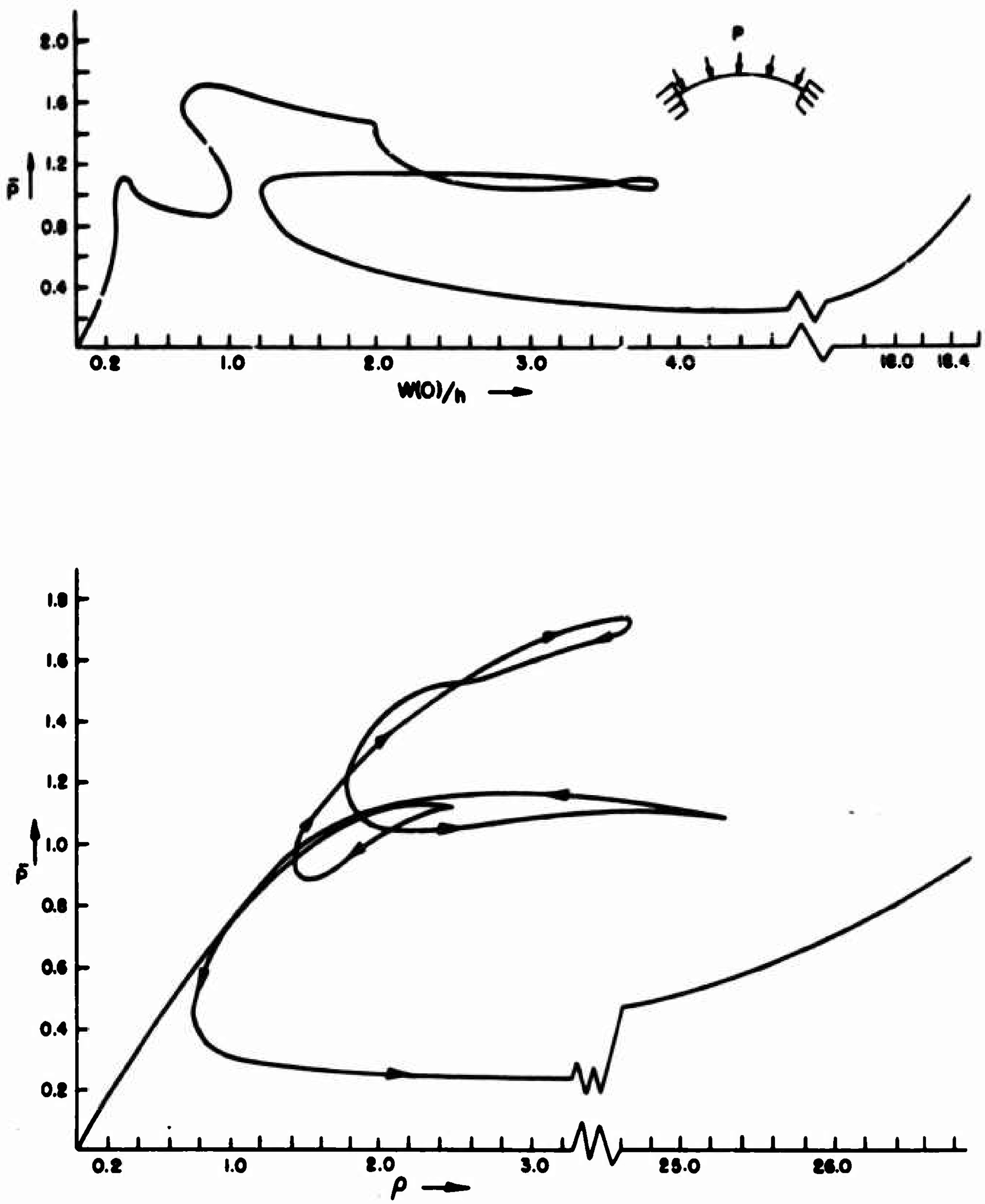

Figure 6. LOAD-DEFLECTION CURVE (CLAMPED EDGE; $\lambda^2 = 64$)

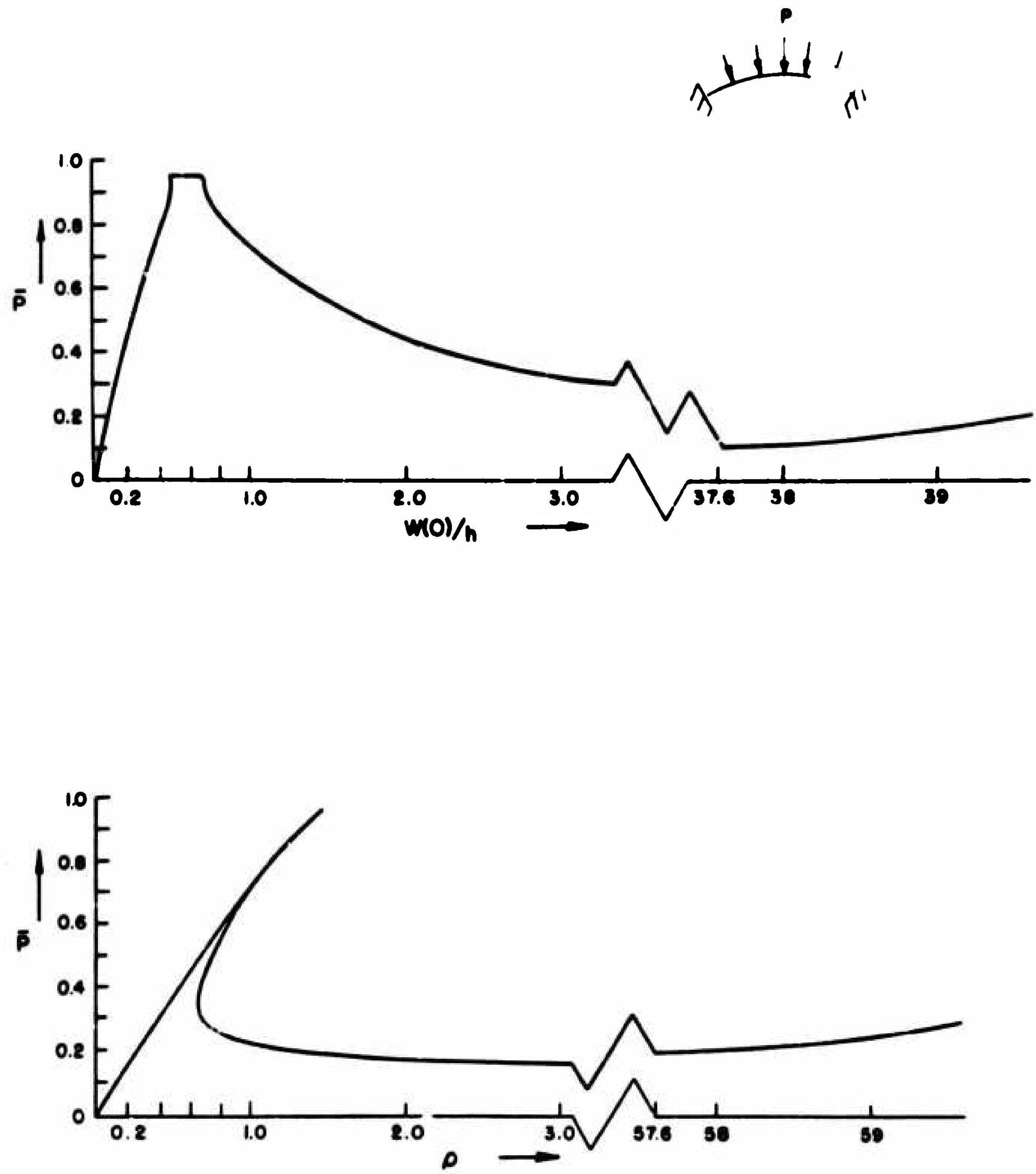

Figure 7. LOAD-DEFLECTION CURVE (CLAMPED EDGE; $\lambda^2 = 144$)

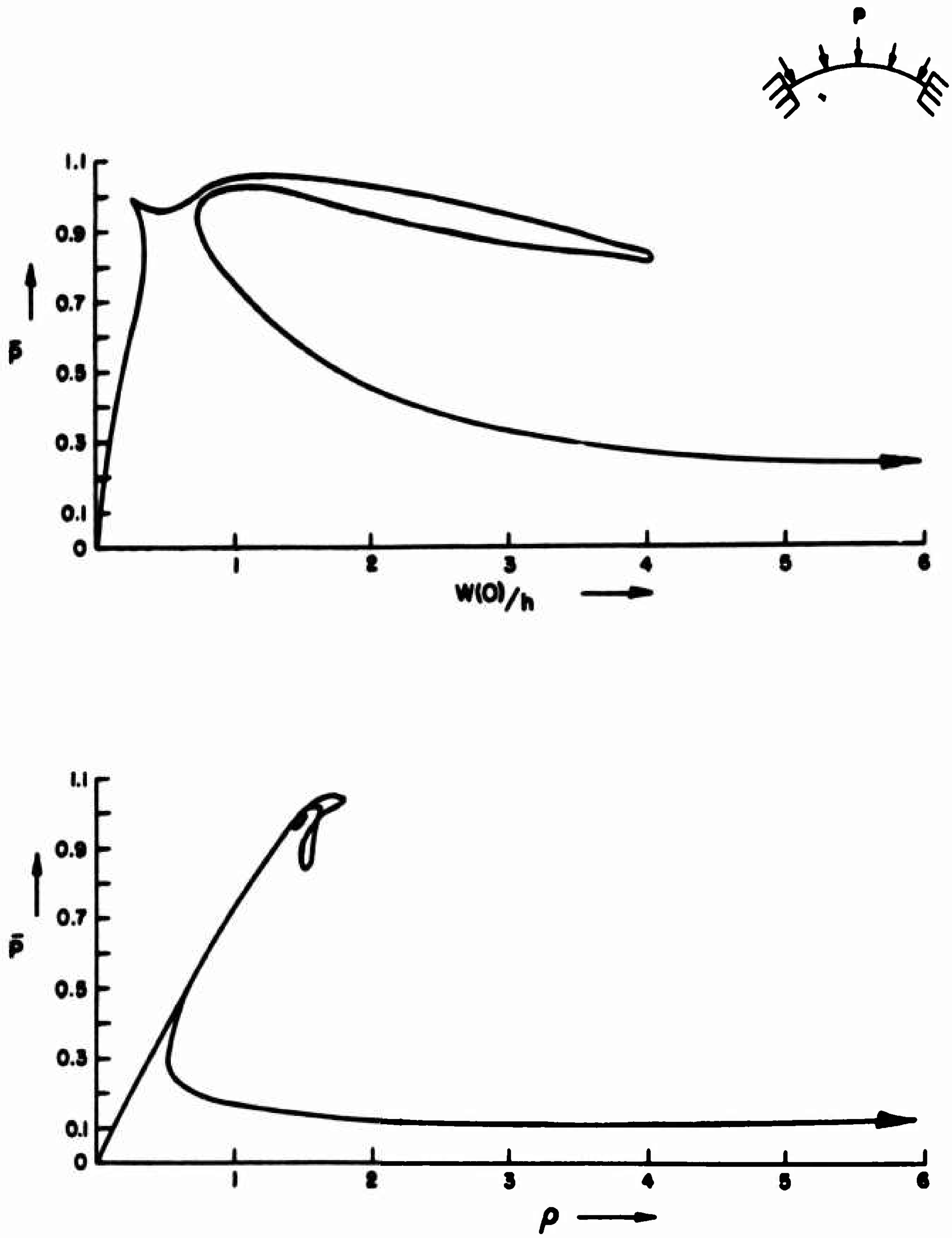

Figure 8. LOAD-DEFLECTION CURVE (CLAMPED EDGE; λ^2 = 400)

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13 ABSTRACT

A numerical procedure for the solution of the nonlinear equations governing the large axisymmetric deflections of thin shells of revolution is presented and applied both to the complete equations due to Reissner and to the simpler equations to which these reduce for small-finite angle changes. Global solutions extending into the postbuckled range are shown to be considerably more complicated than expected. The character of the global solution is also shown to be quite sensitive to boundary conditions imposed. A comparison of the results obtained from the complete equations and the small-finite deflection equations reveals a very close agreement through the entire load-deflection history. (Author)

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14. KEY WORDS	LINK A		LINK B		LINK C	
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